Differential Equations Exam 2 Fall 2018

Name:______________________________

Read This First!

- Show ALL work clearly in the space provided. In order to receive full credit on a problem, solution methods must be complete, logical and understandable.

- Label all graphs as appropriate

- Answers must be clearly labeled in the spaces provided after each question. Cross out or fully erase any work that you do not want graded. The point value of each question is indicated after its statement. No books or other references are permitted.

I attest that I have neither given nor received help of any kind on this exam.

Signature:______________________________

Grading - For Administrative Use Only

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Let \( A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \)

- \( A = a + d \)
- the determinant of \( A = ad - bc \)

\[
J = \begin{bmatrix}
\frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\
\frac{\partial g}{\partial x} & \frac{\partial g}{\partial y}
\end{bmatrix}
\]
1. The following problems refer to a system of two linear differential equations:

\[ \ddot{\vec{u}} = A\vec{u} \]

SEE OTHER ATTACHMENT!
2. Consider the system (let \( g(t) = 0 \)): 

\[
x'' + 3x' - 10x = g(t)
\]

(a) Find the general solution for \( x(t) \). [6]

\[
x(t) = c_1 e^{-5t} + c_2 e^{2t}
\]

(b) Verify that your answer for part (a) is indeed a solution of the above differential equation. (to make this easier just pick half your solution and verify it) [5]

Plug the above \( x(t) \) into the given DE and show that the left side does equal 0.

(c) Consider the initial conditions \( x(0) = 2 \) and \( x'(0) = 0 \). Find any constants from part (a). [5]

\[
c_1 = 4/7 \quad \text{and} \quad c_2 = 10/7
\]

(d) What happens to \( x(t) \) as \( t \to \infty \)? [2]

\[
x \text{ gets infinitely large. } x(t) \to \infty
\]
3. Consider the same system but with \( g(t) = 6 \cos(t) \). Find the general solution for \( x(t) \) 
(you can ignore the initial conditions and leave any constants in your general solution)

\[
x'' + 3x' - 10x = g(t)
\]

\[
x(t) = c_1 e^{-5t} + c_2 e^{2t} - \frac{33}{65} \cos t + \frac{99}{715} \sin t
\]
4. Consider the Lotka-Volterra predator-prey model (let $a = 4, b = 2, c = 1, m = 3$):

\[
\frac{dx}{dt} = ax - bxy \\
\frac{dy}{dt} = cxy - my
\]

(a) Calculate all the null-clines

- x-nullclines: $x = 0$ and $y = 2$
- y-nullclines: $y = 0$ and $x = 3$

(b) Sketch the null-clines below and label any equilibria (you can double check your equilibria algebraically)

Sketch the above horizontal and vertical lines. Equilibria at the origin and at $(3, 2)$. Null clines are vertical and horizontal lines

(c) On the figure above, draw arrows describing the flow in each sector and the direction each nullcline can be crossed.

(d) On the above figure, roughly sketch a sample solution for initial condition $x(0) = 1$ and $y(0) = 1$.

Arrows and sample solution should indicate a ‘center’ with counter-clockwise motion
5. Consider the same Lotka-Volterra predator-prey model (let \( a = 4, b = 2, c = 1, m = 3 \)):

\[
\begin{align*}
\frac{dx}{dt} &= ax - bxy \\
\frac{dy}{dt} &= cxy - my
\end{align*}
\]

(a) Calculate the Jacobian of the above system \[5\]

\[
J = \begin{bmatrix}
4 - 2y & -2x \\
y & x - 3
\end{bmatrix}
\]

(b) Evaluate the Jacobian at each equilibrium \[3\]

\[
J_{0,0} = \begin{bmatrix}
4 & 0 \\
0 & -3
\end{bmatrix}
\]

\[
J_{3,2} = \begin{bmatrix}
0 & -6 \\
2 & 0
\end{bmatrix}
\]

(c) Calculate the eigenvalues at each equilibrium (do NOT calculate the eigenvectors) \[4\]

Origin: \( \lambda = 4, -3 \). \((3, 2)\): \( \lambda = \pm i\sqrt{12} \)

(d) Use the eigenvalues to classify each equilibria (does this classification match what you’d expect from your nullcline analysis?) \[2\]

origin is a saddle and other equilibria is a center